Sovereign Spreads Fluctuations in Emerging Economies: Terms of Trade Matter*

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Abstract

We propose a stochastic general equilibrium model of sovereign default with endogenous default risk in order to explain the interest rate behavior in emerging economies. We incorporate two types of shocks to cover a foreign and a domestic uncertainty. We define as the domestic and the foreign uncertainty, GDP and terms of trade shock, respectively. The model is able to succesfully increase the dispersion of sovereign interest rates when GDP shocks are above the trend. This result seems to suggest that terms of trade is a good candidate to explain the volatility of interest rates in small open economies when they are not under recessions or crises.

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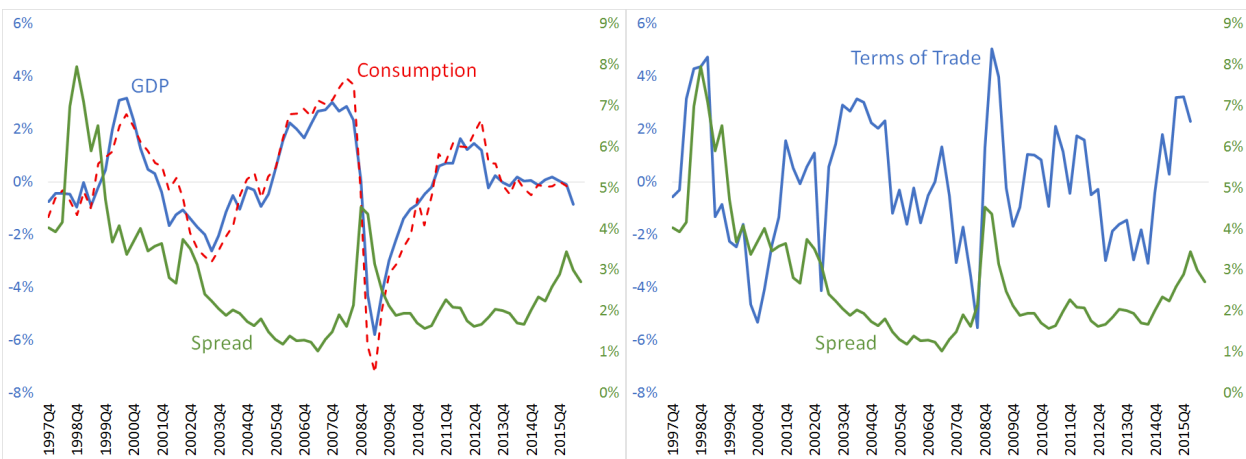
1 Introduction

This paper focuses on the high mean and volatility of interest rate spreads in emerging economies. We document that some emerging economies experience a decrease in the negative correlation between real GDP and interest rate spreads for the last two decades. As a matter of fact, it is observed in the data that interest rates oscillates regardless of a favorable domestic economic performance. Figure 1 shows the Mexican interest rate spread for the last twenty years and its relationship with real GDP and terms of trade. The interest rate spread in Mexico displays sharp rises in 1998 and 2014, even though the economy does not experience deep recessions during these years. Moreover, during these years, the interest rate spread in Mexico follow more closely the movements of the terms of trade series. This observation corresponds to the puzzle proposed in Tomz and Wright (2007). They found that the negative correlation between output and default of a country is remarkably week. Also, they show evidence of countries defaulting on their sovereign debts during good times while making repayments during bad times. This paper addresses this puzzle by arguing that foreign conditions can explain this issue.

Figure 1: Spreads and Cyclical Components in Mexico



The framework presented in Eaton and Gersovitz (1981) is beneficial to analyze the spread behavior because this class of model is able to derive the interest rates endogenously [1]. However, it has been proven difficult to obtain three features of the interest rates with sovereign default models. Arellano (2008) shows countercyclical interest rate spreads by introducing convex costs of default. This yields defaults to be more likely to occur dur-

[1] Aguiar and Gopinath (2007) and Neumeyer and Perri (2005) assume exogenous interest rates since it is useful to explain business cycles of developing countries. High volatility and countercyclicality of interest rates are regarded as crucial parts to explain the cyclical movement of aggregate output and prices.

ing recessions. Nevertheless, the mean spread that the model provides is 3.58% which is relatively low compared to the mean spread of Argentina, which is 10.25%. Also, only few fluctuations of spread are observed with good economic conditions due to the structural features of the probability of default. In fact, the spread generated by the model is almost zero when the country is hit by good endowment shocks. Mendoza and Yue (2011) achieve large volatility of spread in their baseline model by introducing endogenous default costs, but fail also to capture the high mean spread shown in the data. This is because spread and default probabilities are linked to each other directly. Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012) are able to improve the spread behavior in the sovereign default models by incorporating long-duration bonds. This helps to increase mean and standard deviation of spread; however, those studies cannot explain why spreads can be high during good times in the economy. The high volatility of spreads is mainly accomplished by the large dispersion of spreads with low endowments while the standard deviation with high endowments is significantly small.

This paper proposes a stochastic general equilibrium model of sovereign default with endogenous default risk in order to explain the interest rate behavior in emerging economies. The key feature of this paper is that the model incorporate an exogenous foreign shock called terms of trade. In the model, a negative terms of trade shock act in two ways. First, the country spends more in foreign products for consumption. Second, the terms of trade have direct impacts on the level of foreign currency debts that sovereigns owe to foreign lender. This model works with the assumption that sovereigns issue their bonds in foreign prices. As shown in Eichengreen and Hausmann (1999) and Jeanne (2003), emerging economies tend to issue debt in foreign currency because their local currency present high fluctuations and lack of credibility [2] Since debt is issued in foreign currency, the countries are vulnerable to changes in world prices. In other words, when the adverse terms of trade shocks hit the economy, the sovereign immediately encounters an unexpected enlarged debt burden. Consequently, the probability of default is not only affected by countries GDP shocks but also the terms of trade shocks. This provides an explanation of why terms of trade shocks can lead to higher and more volatile spread movements, regardless of the GDP performance.

As mentioned above developing countries experience high volatility of terms of trade and output. More specifically, terms of trade shocks are more volatile than GDP shocks in emerging economies. This implies that the terms of trade shocks are an important factor to consider when studying them. Moreover, as shown in Kose (2002) an important source for the repayment of foreign debt is export revenue and this is largely affected by the terms of trade. Terms of trade are often studied in the sovereign default models. **?**, Gu (2015),

[2] Du and Schreger (2015) show that foreign currency debt composition has decreased since 2004. Nevertheless, they still have a significant level of foreign currency debt.

and Asonuma (2016) endogenously induce the deterioration of the terms of trade and real exchange rate. This paper make distinctions from those papers by assuming terms of trade shocks as exogenous. Popov and Popov, Wiczer et al. (2014) assume an exogenous path of terms of trade but he examines the role of terms of trade penalties and focuses on changes in trade volumes. In contrast, we focus in analyzing the changes in debt burden contingent to terms of trade. Cuadra and Sapriza (2006) study also an exogenous terms of trade shock when the production side buy intermediate imported goods. In their model, the terms of trade shocks are used as if they are productivity shocks so the terms of trade shocks generate real GDP movement. However, this mechanism violates the result in Kehoe and Ruhl (2008) that proves that terms of trade do not have first order effects in real GDP[3]. Moreover, they do not have the convex default costs so the frequency of default generated by the model is unusually small. In this paper we involve both endowment shocks and the terms of trade shocks, while also considering a convex default cost.

The rest of the paper is organized as follows. Section 2 empirical evidence, Section 3 the model, Section 4 quantitative analysis, Section 5 conclusion.

2 Empirical Evidence

2.1 Currency composition of sovereign external debt

In this section, we construct the ratio of foreign currency sovereign external debt to total debt in developing countries. This helps to develop the idea that the terms of trade shocks are of importance to the fluctuations of the economy in emerging economies via foreign currency sovereign debt owed to foreign investors. The definition of external debt is adopted from Du and Schreger (2015). We deviate from their methodology because we are only interested in studying the government debt[4]. Hence, we use the definition of *sovereign external debt* as any debt issued by the government in developing countries and owed to nonresidents, regardless of the market of issuance.

Debt is categorized by three dimensions: issue sector, issue currency, and issue market. Issuance sector is divided into the government and the corporate sector. The debt issued by the central or local governments is counted as government debt while all debt issued by the private sector is regarded as corporate debt. The classification of issue currency is determined by which currency debt is denominated when issued. Local currency (LC) debt refers to debt that is issued in the currency of issuance country while foreign currency

[3]They also show that the terms of trade do not act as a productivity shock in standard models while they do affect real income and consumption in a country.

[4]Their definition of *external debt* includes both public and private debt in order to analyze how default decisions are affected by debt denomination in public and private sectors. However, this paper considers only public debt. Thus, we define *sovereign external debt* instead of *external debt*.

(FC) debt is denominated in another country's currency. Lastly, issuance market is broken down into two markets. When debt is issued under the domestic law inside a country, it is called domestic debt; on the other hand, international debt follows foreign law and issued in international markets. Among these categories, this paper mainly address the combined category of government as issuer sector, foreign currency as issue currency, and both markets as issue market in order to study sovereign external debt.

Bank for International Settlements (BIS) provides amount of outstanding debt data by each classification. However, debt data by debt holder - nonresidents or residents -, which is the main part of definition of *external debt*, are not available. Hence, we follow Du and Schreger (2015) to construct the currency composition of *sovereign external debt*. They make two assumptions for debt holding of nonresidents. First, nonresidents hold all debts in international market, which implies that all international debts are regarded as external debt. Second, nonresidents do not hold any FC debt in domestic market[5]. Based on these two assumptions, the FC sovereign external debt is constructed as follows: amount of outstanding FC debt issued by the government in international market.

Table 1 provides the share of FC sovereign external debt in total sovereign debt. The total sovereign debt is defined as all debts issued by the government so it consists of both domestic debt and external debt. [6] The analysis of this paper is proceed based on sovereign external debt, and it is expected that countries with higher share of external debt denominated in FC are more likely to be exposed by terms of trade shocks. In Table 1, although the substantial heterogeneity for the ratio of FC external debt to total debt is observed, it is sensible that countries are considerably under the influence of it. Moreover, there are some countries that heavily rely on FC debt owed to foreign creditors such as Peru, Argentina, Lebanon, and Lithuania. In particular, the countries that experienced sovereign default events have a tendency to have higher percentage of FC external debt. Argentina had carried on more than 75% of external debt in FC until the default periods and reduced it to approximately 50% in the first half of the 2000's. Peru also has been maintained high share of FC external debt on average. In case of Russia, almost half of the total debt is FC debt owed to nonresidents in 2004 which is significantly large enough to be affected by exchange rate movements.

One of the results from Du and Schreger (2015) is that sovereigns have been using more LC when issuing external debt in government sector so there is a tendency of the decrease in the proportion of FC external debt in total external debt[7]. Nevertheless, analyzing FC external debt is worthy. Since they compare the FC external debt with total external debt, the

[5]They document that the amount of outstanding foreign currency debt in domestic market is notably small so the second assumption is sensible.

[6]The definition of domestic debt in this context is any debt owed to residents within the country.

[7]They analyze $\frac{\text{FC external debt}}{\text{Total external debt}}$ while this paper analyze $\frac{\text{FC external debt}}{\text{External debt+Internal debt}}$. Thus the share of foreign currency external debt in this paper is affect by the amount of debt owe to residents in a country.

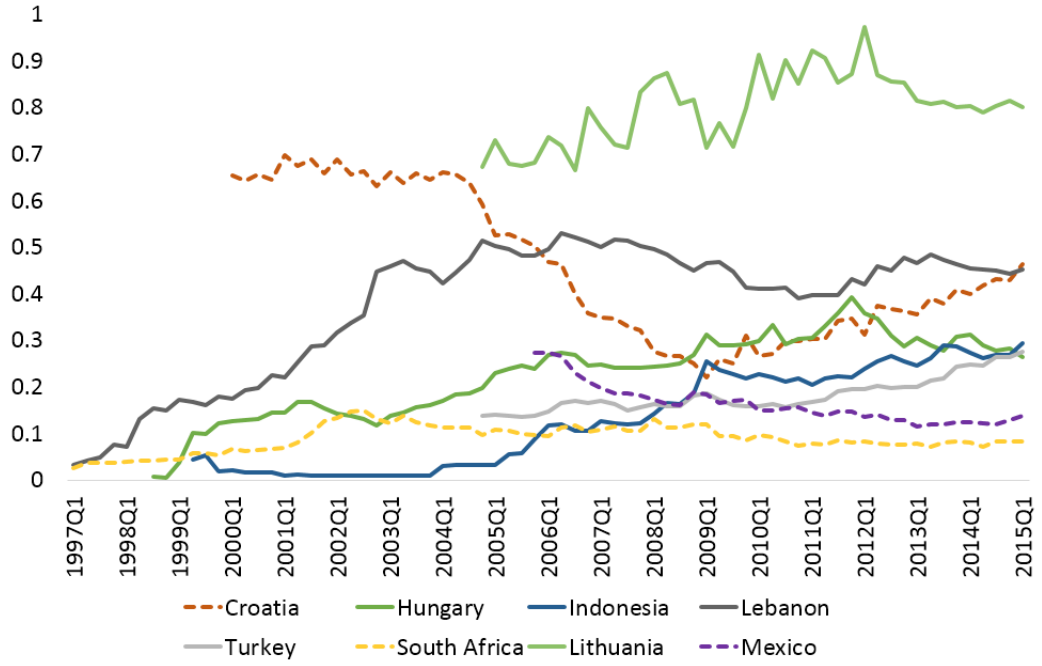
Table 1: Foreign Currency Debt Composition

	Average	2004	2015
Argentina*	79.6	74.9	43.4
Brazil	7.3	4.9	4.4
Chile	12.9	20.8	16.3
Colombia	22.8	22.8	25.5
Croatia	45.7	59.3	46.1
Hungary	22.7	19.8	27.0
Indonesia	13.9	3.2	30.8
Lebanon	35.2	51.5	44.8
Lithuania	83.8	73.2	79.5
Malaysia	3.9	9.0	3.3
Mexico	16.2	27.3	15.1
Peru*	65.1	84.2	38.7
Philippines	24.9	33.9	23.4
Russia*	31.2	48.5	38.1
South Africa	7.2	10.6	9.7
Turkey	19.0	16.5	29.6

Notes: * indicates countries that experienced default events. 2005 data is used for Mexico and Malaysia for the 2004 column and 2007 data and 2008 data are used for South Africa and chile for the 2004 column respectively. They are first year of data availability.

expansion or contraction of the amount of total external debt is not taken into consideration in their construction of currency composition. In other words, the importance of FC external debt could be underestimated if the countries issue more external debt than domestic debt. However, the measure of currency composition used in this paper reflects this issue since the definition of total sovereign debt include both domestic debt and external debt. If the amount of domestic debt gets smaller, then the share of FC external debt in total sovereign debt increases which means external debt in FC becomes a more essential part of the debt in the countries. Actually, all countries except Hungary and Russia in Table 1 display continuous increases in the amount of external debt denominated in FC.[8] Furthermore, it is not explicitly shown that the share of FC external debt has been decreasing in Figure 2 with our data construction. Indonesia, Hungary, Turkey, and Lithuania, for instance, have kept expanding the share of external debt in FC. The LC debt in domestic market rapidly rose in Croatia around 2004 so the share sharply decreased at that time but it started to issue more FC debt in international market in 2009 so the share has been following the growing trend since the time. In addition, other countries hold more or less a constant share of FC external debt. Those empirical evidence illustrates that external debt in FC are still a crucial part of debt in developing countries.

Figure 2: Foreign Debt Profile for Emerging Economies



[8]Hungary and Russia has been reducing the amount of foreign currency external debt since 2014.

3 Terms of trade, GDP and spread across countries

The data presented in Table 2 and 3 are statistics for the terms of trade, GDP, and spread across 24 developing countries. In this paper, the terms of trade (TOT) are defined as the price of imports relative to the price of exports.

$$TOT = \frac{P_M}{P_X}$$

In order to construct the terms of trade, quarterly merchandise customs imports and exports data 1991Q1 to 2015Q4 are obtained from World Bank Global Economic Monitor (GEM). [9] By using current and constant value of import and export each deflator is calculated. Afterwards the import and the export deflators are used for the price of imports and the price of exports respectively. Hence, the terms of trade are constructed by import deflator over export deflator. The quarterly GDP is also provided from GEM and the time period is the same as the one in imports and exports data. The interest rate spread data are taken from J.P. Morgan's EMBI + database.[10] The terms of trade and output are log and HP detrended.

Table 2 provides standard deviation of the terms of trade, GPD, and spread. It also provides the mean of spread in each country since this paper focuses on the behavior of spread. Although there is a cross-country heterogeneity, in almost all sample countries, the standard deviations of TOT is bigger than those of GDP. In other words, TOT is more fluctuate than GDP in most of countries. This fact is crucial for the analysis of the paper since this indicates that more fluctuations of the economy can be driven by the terms of trade shocks with high volatility. Also, defaulted countries, such as Argentina, Ecuador, and Russia, show much higher mean and volatile movement of spread. In particular, the volatility of TOT in Ecuador is approximately 14 times bigger than its GDP. Hence, it can be seen that Ecuador has been affected by volatile terms of trade shocks. However, it is unclear that the terms of trade shocks is a main factor for movement of spread based on magnitude of standard deviation, but correlation with spread will help to improve this issue.

Table 3 shows different combinations of correlations among TOT, GDP and spread in the sample countries. First, the negative correlations between GDP and spread are achieved and also the positive correlations between TOT and spread are presented in most countries. This implies that the deterioration of the terms of trade coincide with the increase in spread in

[9] World Development Indicators (WDI) provides imports and exports data across countries, but this is annual data. Thus, we interpolate annual data to transform to quarterly data and compare it with the quarterly merchandise customs imports and exports and we found that those two series are coincided through the sample period.

[10] The spread data are not a balanced data across countries, so we used series of spread available up to 2015Q4.

Table 2: Descriptive Statistics in Emerging Economies

	σ(TOT)	σ(GDP)	σ(spread)	μ(spread)
Argentina	0.045	0.040	18.24	15.99
Brazil	0.041	0.015	3.92	5.66
Chile	0.090	0.018	0.59	1.46
China	0.036	0.010	0.55	1.18
Colombia	0.073	0.013	2.07	3.56
Dominican Rep	0.022	0.023	3.30	5.39
Ecuador	0.295	0.020	8.38	12.33
Egypt	0.034	0.015	1.74	2.55
Hungary	0.016	0.015	1.58	1.80
Indonesia	0.068	0.033	1.44	2.89
Kazakhstan	0.264	0.024	2.74	4.28
Korea	0.035	0.023	1.04	1.31
Malaysia	0.027	0.018	1.25	1.78
Mexico	0.040	0.023	1.51	2.75
Morocco	0.016	0.013	2.43	2.30
Peru	0.073	0.018	1.96	3.51
Philippines	0.046	0.013	1.52	3.46
Poland	0.096	0.014	0.90	1.71
Russia	0.098	0.029	11.26	7.28
South Africa	0.032	0.012	1.19	2.26
Sri Lanka	0.024	0.009	4.33	6.01
Tunisia	0.014	0.011	0.92	1.84
Turkey	0.030	0.036	2.20	4.02
Ukraine	0.023	0.043	6.23	7.46

Table 3: Correlations in Emerging Economies

	ρ(TOT,spread)	ρ(GDP,spread)	ρ(TOT,GDP)
Argentina	-0.026	-0.647	-0.066
Brazil	0.205	-0.198	-0.553
Chile	0.528	-0.335	-0.478
China	-0.080	-0.007	0.257
Colombia	0.267	-0.197	-0.485
Dominican Rep	0.020	-0.641	-0.006
Ecuador	0.517	-0.445	-0.687
Egypt	0.082	-0.189	0.084
Hungary	0.141	-0.290	0.222
Indonesia	0.574	-0.284	-0.507
Kazakhstan	-0.440	-0.607	0.541
Korea	-0.209	-0.633	0.524
Malaysia	0.061	-0.485	0.116
Mexico	0.443	0.005	-0.466
Morocco	-0.126	0.135	-0.099
Peru	0.332	-0.156	-0.263
Philippines	0.029	-0.326	0.262
Poland	0.007	0.159	0.137
Russia	0.241	-0.487	-0.681
South Africa	0.223	-0.250	-0.021
Sri Lanka	0.433	-0.430	0.229
Tunisia	0.556	-0.162	0.017
Turkey	0.169	-0.449	0.079
Ukraine	-0.194	-0.423	-0.157

general. Moreover, even though it is hard to find a certain pattern between ρ(TOT,spread) and ρ(TOT,spread), there are nine countries having higher correlation of spread with TOT than with GDP. This can be direct evidence for the impacts of TOT on movement of interest rate spread. For example, Mexico have significantly high correlation of spread with TOT while there is almost no correlation with GDP: hence, it is reasonable to conclude that the the fluctuation of Mexican spread is mainly affected by the terms of trade shocks. This interpretation can be generalize to any countries showing higher correlation between spread and TOT.

4 Model

In this section, we propose a model of that incorporates two sources of uncertainty in a country, a domestic and a foreign. We work with a framework that extends the sovereign

default models introduced in Eaton and Gersovitz (1981) and Arellano (2008). We use this last one to incorporate a source of external uncertainty called terms of trade.

Consider a small open economy where there are two types of shocks, a domestic and a foreign. On one hand, the domestic shock is going to be represented by real GDP movements. On the other, the foreign shock is going to be represented by terms of trade movements. Let y_t and p_t represent output and terms of trade in period t, respectively. As discussed previously, it is standard to assume terms of trade as an exogenous variable for small open economies. We consider this in order to construct the following stochastic system

$$\begin{bmatrix} \ln y_{t+1} \\ \ln p_{t+1} \end{bmatrix} = A \begin{bmatrix} \ln y_t \\ \ln p_t \end{bmatrix} + \begin{bmatrix} \varepsilon^y_{t+1} \\ \varepsilon^p_{t+1} \end{bmatrix}, \tag{1}$$

where the roots of the second order polynomial $det\left(I_2 - Ax\right) = 0$ lie outside the complex unit circle and the errors vector is a binormal distribution with mean $\overline{0}$ and variance-covariance matrix Σ. With this formulation we are able to tailor a correlation between contemporanous real GDP and terms of trade.[11]

There are two types of tradable goods in this economy, a domestic and a foreign. In every period, a representative household purchases these two types of goods and transfom them into a final consumption good using the following Armington aggregator technology,

$$C = \left(\lambda c_d^{-\eta} + (1-\lambda) c_f^{-\eta}\right)^{-\frac{1}{\eta}},$$

where C represents the final aggregated good consumption, c_d the domestic good consumption, c_f the foreign good consumption, $\lambda \in [0,1]$ a parameter that captures home bias, and $\eta \in [-1, \infty)$ a parameter that controls the elasticity of substitution between domestic and foreign goods. The representative household is able to purchase these goods taking the output available in the period and an amount of taxes that the government issues in a lump-sum fashion as given. The representative household is a risk averse agent that obtains utility from the stream of final consumption goods obtained in every period as

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u\left(C_t\right)\right],$$

where $\beta \in (0,1)$ is a parameter that captures a discount factor across periods and $u(\cdot)$ is an increasing and strictly concave function.

In this economy there is also a benevolent government whose objective is to maximize the utility of the representative household. In order to achieve this, it has two main decisions to

[11] As shown in Kehoe and Ruhl (2008), it is a common mistake to misrepresent what real GDP is in a model. Moreover, standard models with a production side that buy imported goods at the price of terms of trade capture a spurious correlation between real GDP and terms of trade. If real GDP is measured correctly in them, the correlation between real GDP and terms of trade should be close to zero.

take. The first decision is either to honor or default in its sovereign bonds obligations. If the government takes the decision of honoring them, it has access to international financial markets where it can buy or sell one period maturity sovereign bonds. Then, the second decision is how much of these bonds to purchase given a price schedule contingent to the amount of these new bonds and the shocks the economy experiences. Let B, B', and $q(B', y, p)$ represent the amount of sovereign bonds due, the amount of new sovereign bonds purchases, and the price of these at any given period.

We introduce the assumption that the international financial markets are managed only in foreign goods. This is, the sovereign bonds returns and purchases have to be done in foreign terms. Because of this, the value of sovereign bonds in each period will be subject to the terms of trade shock.

We model two types of penalties for defaulting in sovereign debt, exclusion of financial markets and output losses. With these penalties we want to capture the fact that countries that default in their debt experience a temporary exclusion to international borrowing and periods of poor output performance. We assume that if a government defaults in their debt, this will be erase entirely and it will enter a financial autarkic environment. In addition, the country will experience an output cost that limit its endowment. The economy will remain in this financial autarky for a stochastic number of periods and will re-enter the financial markets with an exogenous probability.

Let x represent exports, the resources exiting the economy. The balance trade condition for the repayment state can be expressed as

$$x - pc_f = pq(B', y, p)B' - pB.$$

The left-hand side is the current account while the right-hand side is the negative of the capital account of the economy. This is, any surplus(deficit) that the capital account experiences due to the government's sovereign debt position will imply a deficit(surplus) of the current account. The balance trade condition for the default state can be expressed as

$$x - pc_f = 0.$$

Considering the exclusion of financial markets penalty of default, the balance trade condition will imply that the current account cannot experience any kind of suplus or deficit.

The resource constraint of the government will show how the domestic good endowment can be consumed in every period. On one hand, if the government chooses to repay its debt, the resource constraint will be

$$c_d + x = y.$$

On the other hand, if the government chooses to default,

$$c_d + x = h(y),$$

where $h(\cdot)$ is an increasing function such that $h(y) \leq y$[12].

There is a representative foreign lender who is able to borrow and lend risk-free bonds at a constant international interest rate $r^* > 0$. We assume it has perfect information about the small open economy. This is, it can observe the level of output and terms of trade that the small open economy experiences every period. The foreign lender is risk neutral[13] and maximizes expected profits over risky sovereign bonds from the small open economy. We assume that the foreign lender maximizes profit only in foreign goods terms. Let δ represent the probability of the government to default in its sovereign debt position. Taking as given the default probability and the bond price, the foreign creditor chooses $\tilde{B}'$ to maximizes

$$\underset{\tilde{B}'}{\text{Max}} \left\{ q\tilde{B}' - \left(\frac{1-\delta}{1+r^*} \right) \tilde{B}' \right\}.$$

By the risk neutral nature of the foreign lender, the bond price schedule solution in equilibrium must satisfy the first order condition of the previous problem. The bond price schedule will be a result of the following break-even condition,

$$q = \frac{1-\delta}{1+r^*}. \tag{2}$$

Realising that $\delta \in [0,1]$, we can infer that $q \in \left[0, \frac{1}{1+r^*}\right]$. Defining the sovereign interest rate as $r \equiv \frac{1}{q} - 1$, we obtain that $r = [r^*, \infty)$. Finally, we define the sovereign bond spread as $S \equiv r - r^*$.

The timing of the government problem is the following. In the beginning of the period the government realises the amount of sovereign bonds due B, the domestic goods endowent shock y, and the terms of trade shock p. The government asses the optimal relationship between domestic and foreign goods using p and the preferences of the representative household. Then the government decides whether to honor or default in its debt obigations. If it decides to honor its debt, the government updates its sovereign bond holdeings B' taking as given the sovereign bond price schedule $q(B', y, p)$ and constrained to its resource constraint and the balance trade condition. The foreign lender takes the bond price q as given and supplies $\tilde{B}'$ matching B'. Purchases of foreign and domestic take place. Finally, the representative household consumes the final good by aggregating the domestic and foreign goods.

[12] Realise that the output cost is defined as $y - h(y)$ and is non-negative by definition.

[13] Cole and Kehoe (1996) explains that the risk neutrality of the foreign lenders reflect the fact that the size of an individual sovereign transaction is relatively small compared to the total international credit market.

4.1 Recursive Equilibrium

There are three variables that define the state of the government in every period: the sovereign bonds due, the output shock, and the terms of trade shock. Define $V(B, y, p)$ as the value function of the government at the beginning of every period. Let us model the first decision of the government as

$$V(B, y, p) = \text{Max}\left\{V^R(B, y, p), V^D(y, p)\right\}, \tag{3}$$

where $V^R(B, y, p)$ and $V^D(y, p)$ represent the value of the government if it repays and defaults in its debt obligations, respectively. Here, the government chooses which environment will yield the highest welfare for the representative household captured by $V^R(B, y, p)$ and $V^D(y, p)$.

When the government chooses to repay, it chooses the amount of sovereign bonds to sell or purchase as well as the household allocations that will maximize its welfare subject to the resource constraint, the balance trade condition, the aggregation technology, and a no-Ponzi condition. In order to do this, it takes the price schedule for the bonds as given and a lower boundary for the sovereign bonds issuance $\mathbb{B} > 0$. Thus,

$$\begin{aligned} V^R(B, y, p) = \underset{x, c_d, c_f, C, B'}{\text{Max}} & \left\{u(C) + \beta \mathbb{E}_{(y', p')}\left[V(B', y', p')|(y, p)\right]\right\} && (4) \\ \text{s.t. } & c_d + x = y && \text{(Resource Constraint)} \\ & x - pc_f = pq(B', y, p)B' - pB && \text{(Balance Trade Condition)} \\ & C = \left(\lambda c_d^{-\eta} + (1-\lambda)c_f^{-\eta}\right)^{-\frac{1}{\eta}} && \text{(Aggregation Technology)} \\ & B' \geq -\mathbb{B}. && \text{(No-Ponzi Condition)} \end{aligned}$$

When the government chooses to default, it chooses the household allocations that will maximize its welfare subject to the resource constraint, the balance trade condition, and the aggregation technology. In order to do this, it takes the default penaltiy and the probability of returning to the financial markets $\phi \in [0, 1]$ as given. Recall that the default penalties are the output costs and the zero current account restriction due to the financial market exclusion. Thus,

$$\begin{aligned} V^D(y, p) = \underset{x, c_d, c_f, C}{\text{Max}} & \left\{u(C) + \beta \mathbb{E}_{(y', p')}\left[\phi V(0, y', p') + (1-\phi)V^D(y', p')|(y, p)\right]\right\} && (5) \\ \text{s.t. } & c_d + x = h(y) && \text{(Resource Constraint)} \\ & x - pc_f = 0 && \text{(Balance Trade Condition)} \\ & C = \left(\lambda c_d^{-\eta} + (1-\lambda)c_f^{-\eta}\right)^{-\frac{1}{\eta}}. && \text{(Aggregation Technology)} \end{aligned}$$

In order to define what is the probability of default for a government, it is useful to characterize the set of output and terms of trade states in which a government finds optimal to default contingent to a level of sovereign bond holdings. Specifically, define the default set as the

$$\mathcal{D}(B) = \left\{(y,p) \in \mathbb{R}^2_{++} : \quad V^D(y,p) > V^R(B,y,p)\right\}. \tag{6}$$

This set expresses that if the government sells B' and the shocks of next period are $(y',p') \in \mathcal{D}(B')$, then the government will find it optimal to default on B' next period. Because of this, we can define the probability for a government to default on B' by measuring how likely is to end up in the states that live in $\mathcal{D}(B')$. Given the stochastic process that govern the movements of output and terms of trade shocks, call $f(\cdot)$ the probability density function between shock states. Thus, the probability of default can be expressed as

$$\delta(B',y,p) = \int_{(y',p')\in\mathcal{D}(B')} f\left((y',p')|(y,p)\right) d(y',p'). \tag{7}$$

Consider the case where the government chooses a level of sovereign bonds B' such that there are no possible states in the next period in which it will default on them, $\mathcal{D}(B') = \emptyset$. Then, with that amount of sovereign bonds the probability of default will be zero, $\delta(B',y,p) = 0$. Also, consider the case when the government chooses a level of sovereign bonds B' such that default is for sure regardless of the shock realisations in the next period, $\mathcal{D}(B') = \mathbb{R}^2_{++}$. Then, with that amount of sovereign bonds the probability of default will be one, $\delta(B',y,p) = 1$. Finally, realise that if the output and terms of trade shocks have no persistency whatsoever, then the probability of default will only be contingent in the sovereign bond issues or purchases.

Now, the bond price schedule must satisfy the break-even condition (2) from the representative foreign lender problem. Considering the probability of default, the representative foreign lender must be consistent to (7). Thus, the bond price schedule will be

$$q(B',y,p) = \frac{1-\delta(B',y,p)}{1+r^*}. \tag{8}$$

Realise that this break-even condition will yield zero profit in expectation to the representative foreign lender regardless of the quantity of sovereign bonds it purchases or sells $\tilde{B}'$. Therefore, it will be willing to cover any amount of sovereign bonds the government finds optimal to choose during the repayment state. In other words, the sovereign bond market always clear in equilibrium $\tilde{B}' = B'$.

Definition 1 (Recursive Equilibrium) *The recursive equilibrium of this small open economy will be a set of government value functions $V(B,y,p)$, $V^R(B,y,p)$, $V^D(y,p)$ and a sovereign bonds policy rule $\hat{B}'(B,y,p)$, a set of household consumption policy rules $\hat{c}_d(B,y,p)$,*

$\hat{c}_f(B,y,p)$, and $\hat{C}(B,y,p)$, an exports policy rule $\hat{x}(B,y,p)$, a default set $\mathcal{D}(B,y,p)$, a default probability schedule $\delta(B,y,p)$, and bond price schedule $q(B,y,p)$ such that the following conditions are satisfied:

- *Benevolent Government (Initial)*
  The default set $\mathcal{D}(B,y,p)$ is consistent with the set of government value functions $V(B,y,p)$, $V^R(B,y,p)$, and $V^D(y,p)$ and solves (3).

- *Benevolent Government (Repayment)*
  If $(y,p) \notin \mathcal{D}(B)$ and taking as given the bond price schedule $q(B,y,p)$, the government chooses the sovereign bonds policy rule $\hat{B}'(B,y,p)$, the set of household consumption policy rules $\hat{c}_d(B,y,p)$, $\hat{c}_f(B,y,p)$, and $\hat{C}(B,y,p)$, and the exports policy rule $\hat{x}(B,y,p)$ in order to solve (4) and its solution is consistent with $V^R(B,y,p)$.

- *Benevolent Government (Default)*
  If $(y,p) \in \mathcal{D}(B)$, the government chooses the set of household consumption policy rules $\hat{c}_d(B,y,p)$, $\hat{c}_f(B,y,p)$, and $\hat{C}(B,y,p)$, and the exports policy rule $\hat{x}(B,y,p)$ in order to solve (5) and its solution is consistent with $V^D(y,p)$.

- *Default probability*
  The default probability schedule $\delta(B,y,p)$ is consistent with the default set $\mathcal{D}(B,y,p)$ and (7).

- *Bond Pricing*
  The bond pricing schedule $q(B,y,p)$ is consistent with the probability of default schedule $\delta(B,y,p)$ and (8).

4.2 Aggregate Recursive Equilibrium

The problem described in the previous section has an intratemporal condition between consumption of foreign and domestic goods by the representative household. Specifically, this intratemporal condition balances the terms of trade with the marginal rate of substitution between foreign and domestic consumption goods,

$$p = \frac{u'(C) \cdot \frac{\partial C}{\partial c_f}}{u'(C) \cdot \frac{\partial C}{\partial c_d}} = \left(\frac{1-\lambda}{\lambda}\right)\left(\frac{c_d}{c_f}\right)^{1+\eta}. \tag{9}$$

Using the final consumption aggregator and (9), let us define the final consumption price index as

$$\mathcal{P}(p) \equiv \left(\lambda^{\frac{1}{1+\eta}} + (1-\lambda)^{\frac{1}{1+\eta}} p^{\frac{\eta}{1+\eta}}\right)^{\frac{1+\eta}{\eta}}. \tag{10}$$

This price lets us weight the price of the final consumption good in the economy considering how important are domestic and foreign goods in its aggregation. Notice that the limit expression of home bias follow,

$$\lim_{\lambda \to 0} \{\mathcal{P}(p)\} = 1 \qquad \text{and} \qquad \lim_{\lambda \to 1} \{\mathcal{P}(p)\} = p.$$

On one hand, if there is complete home bias, the price index of the final consumption good is not affected at all by the terms of trade. This result is intuitive because it tells us that the representative household does not derive any utility from foreign consumption goods. On the other hand, if there is complete foreign bias, the price index of the final consumption good is the complete terms of trade price.

Consider an aggregate version of the government's problem when it chooses to repay,

$$V^R(B,y,p) = \underset{C,B'}{\text{Max}} \left\{ u(C) + \beta \mathbb{E}_{(y',p')} \left[V(B',y',p') | (y,p) \right] \right\} \tag{11}$$

$$\text{s.t. } \mathcal{P}(p)C + pq(B',y,p) = y + pB \qquad \text{(Resource Constraint)}$$

$$B' \geq -\mathbb{B}. \qquad \text{(No-Ponzi Condition)}$$

Also, consider the an aggregate version of the government's problem when it chooses to default,

$$V^D(y,p) = \underset{c_d,c_f,C}{\text{Max}} \left\{ u(C) + \beta \mathbb{E}_{(y',p')} \left[\phi V(0,y',p') + (1-\phi) V^D(y',p') | (y,p) \right] \right\} \tag{12}$$

$$\text{s.t. } P(p)C = h(y) \qquad \text{(Resource Constraint)}$$

Definition 2 (Aggregate Recursive Equilibrium) *The aggregate recursive equilibrium of this small open economy will be a set of government value functions $V(B,y,p)$, $V^R(B,y,p)$, $V^D(y,p)$ and a sovereign bonds policy rule $\hat{B}'(B,y,p)$, a household final consumption policy rule $\hat{C}(B,y,p)$, a final consumption good price index $\mathcal{P}(p)$, a default set $\mathcal{D}(B,y,p)$, a default probability schedule $\delta(B,y,p)$, and bond price schedule $q(B,y,p)$ such that the following conditions are satisfied:*

- *Benevolent Government (Initial)*
  The default set $\mathcal{D}(B,y,p)$ is consistent with the set of government value functions $V(B,y,p)$, $V^R(B,y,p)$, and $V^D(y,p)$ and solves (3).

- *Benevolent Government (Repayment)*
  If $(y,p) \notin \mathcal{D}(B)$ and taking as given the bond price schedule $q(B,y,p)$ and the final consumption good price index $\mathcal{P}(p)$, the government chooses the sovereign bonds policy rule $\hat{B}'(B,y,p)$ and the household final consumption policy rule $\hat{C}(B,y,p)$ in order to solve (4) and its solution is consistent with $V^R(B,y,p)$.

- *Benevolent Government (Default)*
  If $(y,p) \in \mathcal{D}(B)$ and taking as given the final consumption good price index $\mathcal{P}(p)$, the government chooses household final consumption policy rules $\hat{C}(B,y,p)$ in order to solve (5) and its solution is consistent with $V^D(y,p)$.

- *Price Index*
  The final consumption good price index follows (10).

- *Default probability*
  The default probability schedule $\delta(B,y,p)$ is consistent with the default set $\mathcal{D}(B,y,p)$ and (7).

- *Bond Pricing*
  The bond pricing schedule $q(B, y, p)$ is consistent with the probability of default schedule $\delta(B, y, p)$ and (8).

Proposition 3 (Recursive Equilibrium Isomorphism) *The equilibriums defined in Definition 1 and Definition 2 are isomorphic.*

Proof See Appendix A.

This transformation of the original problem is very useful for solving the model and to understand how terms of trade shocks work in our environment. Firstly, realise that the final consumption good price index is non-decreasing[14] in terms of trade regardless of the parameters λ and η,

$$P'(p) = \left(\frac{(1-\lambda)P(p)}{p}\right)^{\frac{1}{1+\eta}} \geq 0.$$

This implies that, regardless of the complementarity or substitutaility of the domestic and foreign goods, the final consumption good price index will keep a monotonic behaviour throughout all the domain of the terms of trade. Moreover, the final consumption good price index will work as a shock absorber of the terms of trade shock. This is, the households will only experience a fraction of the terms of trade shock in terms of final good expenditure. Therefore, terms of trade will have two main effects in the model. The first effect is adjsuting the price of final consumption goods. The second effect is expanding or contracting the debt burden of soverign bonds. This last effect is present under the assumption that the sovereign government issues debt in foreign currency.

Proposition 4 (Default Sets Monotonicity) *Pick an arbitrary level of sovereign bonds B_1 such that $\mathcal{D}(B_1) \neq \emptyset$, if $B_2 \leq B_1$ then $\mathcal{D}(B_1) \subseteq \mathcal{D}(B_2)$.*

Proof See Appendix A.

This result is originally taken from Eaton and Gersovitz (1981), Arellano (2008), and Chatterjee, Corbae, Nakajima and Ríos-Rull (2007) and it is common in sovereign default models. This result tells us that incentives are monotonic with respect of sovereign bonds. Hence, it is mainly followed because the bond pricing $q(B', y, p)$ is independent from the sovereign bonds due in a period. In our model, this relationship does not exist because of the assumption of risk neutrality from the representative foreign lender. Thus, given a level of output and terms of trade, the sovereign bonds due in the period acts only as a shifter in the available amount of resources in the economy. Therefore, if for a level of sovereign bonds

[14] Furthermore, it is strictly increasing as long as we assume there is no full home-bias in the model $\lambda \in [0, 1)$

a government finds optimal to default, then for a lower level of soverign bonds the default decision will still be optimal because the government will have less resources overall.

Using Proposition 4 we can conclude that the bond pricing $q(B, y, p)$ is non-decreasing in sovereign bonds. Let us focus in the case where the government borrows resources from foreign lenders. As the government increases the amount of borrowing, the actual amount of resources received will decrease because the bond pricing contracts. This resources reduction of borrowing compensates the default probability that the government can incur in the following period.

4.3 No Persistency Case

Let us study the case in which the stochastic system proposed in (1) has no persistency, therefore it is i.i.d. binormal distribution with mean $\overline{0}$ and variance-covariance matrix Σ. In this case, the probability of default and the sovereign bond price schedule lose their contingency with respect of GDP and terms of trade. This is because current levels of GDP and TOT do not provide any information about the future realizations of them. We assume also that there are no penalty costs, $h(y) = y$; and default is permanent, $\phi = 0$.

Proposition 5 (No Resources Inflows) *For every sovereign bonds B such that $\mathcal{D}(B) \neq \emptyset$, every feasible B' will yield no resources inflows, $q(B')B' - B \geq 0$.*

Proof See Appendix A.

Default episodes happen when governments are unable to roll-over their debt. This idea is captured with Proposition 5. If the government is in a state where he chooses to default, all the feasible issuances of sovereign bonds must have not been enough to cover the sovereign bonds due in the period. Specifically, there was no feasible issuances of sovereign bonds that could have given a positive flux of resources from outside, $B - q(B')B' \not> 0$.

Proposition 6 (GDP Default Incentives) *Pick an arbitrary level of terms of trade p and sovereign bonds B such that $\mathcal{D}(B) \neq \emptyset$, if $y_2 \leq y_1$ and $(y_1, p) \in \mathcal{D}(B)$ then $(y_2, p) \in \mathcal{D}(B)$.*

Proof See Appendix A.

Default episodes also happen when economies experience recessions, periods where GDP levels are below the trend. Proposition 6 is able to capture this idea. This shows that there if a government finds optimal to default on a level of sovereign bonds for a given recession, any deeper recession would make it default as well. This result is driven mainly because the country will be poorer and there are not contracts available that provide an influx of resources from abroad.

Proposition 7 (Terms of Trade Default Incentives) *Pick an arbitrary level of GDP y and sovereign bonds B such that $\mathcal{D}(B) \neq \emptyset$, if $p_2 \geq p_1$ and $(y, p_1) \in \mathcal{D}(B)$ then $(y, p_2) \in \mathcal{D}(B)$.*

Proof See Appendix A.

We have a similar result for the case of terms of trade rising higher than the trend. Proposition 7 is able to capture this idea. This shows that there if a government finds optimal to default on a level of sovereign bonds for a level of terms of trade, any higher level of terms of trade would make it default as well. This result is driven mainly because the country will experience more expensive goods from the exterior and the resources that flow outside the country will grow because of the foreign good coversion.

5 Quantitative Analysis

In this section, we study Mexico and its business cycles statistics. Also, we describe the calibration process of the model fitting the parameters to the Mexican economy. We solve the aggregate recursive equilibrium described in Definition 2 applying a value function iteration process using a grid search method. The complete algorithm can be found in Appendix B.

5.1 Data

Let us study the Mexican business cycle behavior as an emerging economy. Using OECD Statistics, we obtain quarterly data seasonally adjusted at quarterly levels, and at current and constant prices from 1993Q1 to 2016Q2 for the series of *gross domestic product*, *private final consumption expenditure*, *exports of goods and services*, and *imports of goods and services*. Also, using Global Financial Database, we obtain quarterly data from 1997Q4 to 2016Q3 for the series of *Emerging Markets Bond Index (EMBI+)*. The series of EMBI+ is provided by J.P. Morgan and portrays the long-term spread between yields from sovereign bond and the U.S. Treasuries. We construct the terms of trade series as the ratio of imports price delfator and exports price deflator following Kehoe and Ruhl (2008) methodology[15]. We apply the *HP-filter* with a smoothing parameter of 1600 to the real consumption, real output and terms of trade log-series in order to obtain the cyclical components of them. In addition, we compute the ratio of the difference between exports and imports, over GDP to construct the series of trade balance.

Table 4 shows the business cycles statistics for the Mexican economy. The table shows regular characteristics of emerging economies shown in Neumeyer and Perri (2005). The

[15]There are other ways to compute terms of trade. For example, Mendoza (1995) uses the ratio of exports and imports volumes. We choose not to use this methodology for convenience. Establishing terms of trade as the ratio of imports and exports deflator matches closely the movements of real exchange rates.

Table 4: Business Cycles Statistics for Mexico

Variable	μ	σ	$\rho(\cdot$,Spread)	$\rho(\cdot$,GDP)	$\rho(\cdot$,TOT)
Spread	2.71%	1.47%	-	0.0352	0.2178
GDP	-	2.22%	0.0352	-	-0.4480
Terms of Trade	-	2.56%	0.2178	-0.4480	-
Consumption	-	2.65%	-0.0367	0.9518	-0.4673
Trade Balance	-0.84%	1.62%	0.3335	-0.4819	0.1501

volatility of consumption is higher compared to the volatility of GDP. Another important characteristic is that trade balance is countercyclical. Nevertheless, we are not able to find that interest rates (captured by the sovereign spread series) are countercyclical after 1997Q4[16].

Terms of trade also play an important feature. As documented in Mendoza (1995), Kose (2002), and Broda (2004), there is a strong negative relationship between terms of trade and real GDP in emerging countries. Also, the volatility of terms of trade is higher compared to the volatility of GDP. An important feature we are able to provide is that the correlation between terms of trade and the spreads is significantly higher than the correlation between GDP and spreads. This seems to suggest that the movements the Mexican experienced after 1997 may be better explained by movements in terms of trade rather than movements in real GDP.

5.2 Calibration

We use the real GDP and terms of trade *HP-filter* log-series in order to estimate (1). We assume that the errors vector components are independent from each other[17] and distributed as $\varepsilon^y_{t+1} \sim N(0, \sigma^2_y)$ and $\varepsilon^y_{t+1} \sim N(0, \sigma^2_y)$. We estimate the matrix A using standard OLS

[16] We analyzed this issue further. We use the database provided by Neumeyer and Perri (2005) and construct an implied EMBI+ series from 1994Q1 to 1997Q3. In the database the authors provide the interest rate of several emerging economies from 1994Q1 to 2002Q2. We take the series of Mexican interest rates and substract the yield of US 10-year Treasury constant maturity. We find that the constucted EMBI+ series resembles closely the original EMBI+ series in the quarters they overlap with the exception of the period 2001Q2-2002Q2. Filling the missing quarters for the spread series we are able to find that the interest rates (captured by the sovereign spread series) is countercyclical, with a correlation of -0.4275.

[17] We make this assumption for simplicity. Nevertheless, we are expecting in relaxing this. The error of the real GDP regression surely is correlated with contemporanous terms of trade. In this way, the assumption of the errors being independent should be worked upon.

regressions[18],

$$A = \begin{bmatrix} 0.8399 & -0.0291 \\ -0.1466 & 0.5367 \end{bmatrix}.$$

Once estimated the matrix A, we recover the observed errors and estimate the standard deviations of the errors. The standard deviations of the GDP and terms of trade errors are $\sigma_y = 0.0105$ and $\sigma_p = 0.0203$, respectively. Finally, we discretize the 2-dimensional VAR(1) process into a 289 Markov chain (17 GDP and 17 terms of trade shock levels) using a quadrature method algorithm following Tauchen and Hussey (1991) with a 3 standard deviations mean centered bandwidth.

We use a standard CRRA utility function to convey the representative household preferences,

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma},$$

where σ represents the constant relative risk aversion parameter. We settle this risk aversion parameter to the value of 2. This is a common value in the international real business cycles literature.

During default episodes we impose GDP penalty via the increasing function $h(\cdot)$. An important issue to address when modeling this is its sensitivity contingent to the state of the economy. We use Arellano (2008) convex GDP cost formulation,

$$h(y) = \begin{cases} y & \text{if } y \leq \kappa\mathbb{E}[y] \\ \kappa\mathbb{E}[y] & \text{if } y > \kappa\mathbb{E}[y] \end{cases},$$

where $\kappa > 0$ is a contraction of the long-run mean of GDP. This formulation has the advantage of making default less sensitive to GDP shocks. In particular, defaulting with a GDP level below the threshold of $\kappa\mathbb{E}[y]$ there is no GDP penalty. Nevertheless, above this threshold the GDP penalty increases the higher GDP is.

We use the Global Financial Database in order to obtain the series of *US 10-year Treasury constant maturity yield*. We pick a 10-year Treasury maturity bond because the EMBI+ series relies on long-term maturity bonds. We fix the risk-free interest rate as the average yield from 1997Q1 to 2016Q3, which is 1.62%.

The literature show a wide variety of possible elasticity of substitution between domestic and foreign goods. As noted by Ruhl (2008), this elasticity can be small to account the quarterly fluctuations in trade balances and terms of trade, or high to account the growth

[18] We make this assumption to capture a higher correlation between contemporanous real GDP and terms of trade. Terms of trade can be considered as exogenous for small open economies as motivated in Broda (2004). Therefore, previous real GDP affecting contemporaneous terms of trade should be restricted.

in trade due to trade liberalization. In Kose, Towe and Meredith (2004) they propose this elasticity of substitution to be 1.05 for Mexico when analyzing the NAFTA effect on trade. We choose not to use this level due to the different time span of study we are interested. Nevertheless, we find that other updated papers for the Mexican economy have a similar elasticity of substitution that considers the trade liberalization that Mexico has experienced in the past decades. We use the elasticity of substitution between domestic and foreign goods presented in Cuadra and Nuguer (2016) to calibrate η. They use an elasticity of subtitution of 1.5556, which implies a parameter η of -0.3571.

We use the intratemporal condition (9) in order to calibrate the home bias parameter. Realise that the intratemporal condition can be rewritten as

$$\left(\frac{c_f}{C}\right)^{1+\eta} = \frac{(1-\lambda)\mathcal{P}(p;\lambda,\eta)}{p} \tag{13}$$

In order to construct the series of consumption of foreign goods, we obtain from the Mexican central bank the annual share of imported consumption goods from total imports from 1997 to 2015. We find that the average share is of 13.18% during this period[19]. Using *total imports of goods and services*, *private final consumption expenditure*, and terms of trade quarterly series, the parameter value of $\eta = -0.3571$, the fixed share of imported consumption goods, and the final consumption good price index (10) formula; we find a series of the home bias λ_t that solves in every quarter the intratemporal condition (13). We find that the average home bias is of 0.8748 from 1997Q4 to 2016Q2.

We calibrate the default GDP penalty and the discount factor in order to match two moments of the Mexican economy. Mexico has defaulted in its sovereign debt twice (1928 and 1982) in the last hundred years[20]. This gives a rough estimate of a 2% default probability. We then focus the targets to be this default probability and the standard deviation of trade balance over GDP ratio shown in Table 4. Finally, we keep the probability of re-entry to financial markets proposed by Arellano (2008)[21]. Table 5 presents the parameters

[19] We also find that from 2007 to 2015 this share almost doubled from 8.5% to 14.24%. We also find that most of this increase happened in the first six years of the sample. For this reason, we consider this share as a constant for the calibration process.

[20] As noted in Reinhart and Rogoff (2009), the default years were when two important international crises happened, the *Great Depression* and the *Emerging Markets Crises*. During the *Tequila Crisis* in 1994, Mexico was close to default with its international lenders. Thanks to the internatinal help from the USA, Mexico was able to have dodge this.

[21] In Gelos, Sahay and Sandleris (2011), it is shown that the average waiting period for re-entry after a default has decreased significantly. They show that this average fell from 5 years in the 1980's to 1.6 years in the 1990's. Nevertheless, in Alessandro et al. (2011), they conclude that this comparison between decades is not fair. In particular, that the decrease in average waiting period for re-entry has not decreased throughout time. They provide evidence that, if re-entry to to financial markets do not happen after three years of the default, it is significantly harder to achieve this re-entry. This seems to suggest that the probability of re-entry has not experienced important movements throughout the years.

specification from the calibration and estimation strategy.

Table 5: Parameters Specification

Parameter	Value	Source
A	$\begin{bmatrix} 0.8399 & -0.0291 \\ -0.1466 & 0.5367 \end{bmatrix}$	OLS estimators
σ_y	0.0105	Observed errors
σ_p	0.0203	Observed errors
σ	2.00	IRBC Literature
r^*	1.62%	US 10-year Treasury
η	-0.3571	Cuadra and Nuguer (2016)
λ	0.8748	Intratemporal Condition
β	0.9530	Default probability
κ	0.9690	Trade balance volatility
ϕ	0.2820	Arellano (2008)

5.3 Results

We use a value function iteration process using a grid search method[22] to solve the model. To study the policy rules we establish a high and low level of GDP and terms of trade as $\pm 0.0489\%$ and $\pm 0.0467\%$ deviations from their means, respectively. We then do a 100 simulation processes for the economy for 10,000 periods, starting with zero sovereign bond holdings and in the long-run level of GDP and terms of trade. We burn the first 500 periods of each simulation and compute the business cycles statistics of the model as the averages of the simulations.

Figure 3 shows the boundary limits of the default set (6). Because of Proposition 4 we know that these sets are monotonic with respect of the level of sovereign bond holdings. The left panel shows the relationship with respect of GDP fixing the terms of trade level. The default set is the area south west of the boundaries. We find that higher levels of sovereign debt increases the levels of GDP inside the the default set. In addition, having a higher terms of trade level increases slightly the levels of GDP inside the default set. The right panel shows the relationship with respect of terms of trade fixing the GDP level. The default set is the area north west of the boundaries. We find that higher levels of sovereign debt increases the levels of terms of trad inside the default set. In addition, having a higher GDP levles decreases significantly the levels of terms of trade inside the default set.

[22] We use this method as a first approach to solving the equilibrium of the model. As Hatchondo, Martinez and Sapriza (2010) shows, grid search methods can give spurious results in the business cycle analysis.

Figure 3: Default Set Boundaries

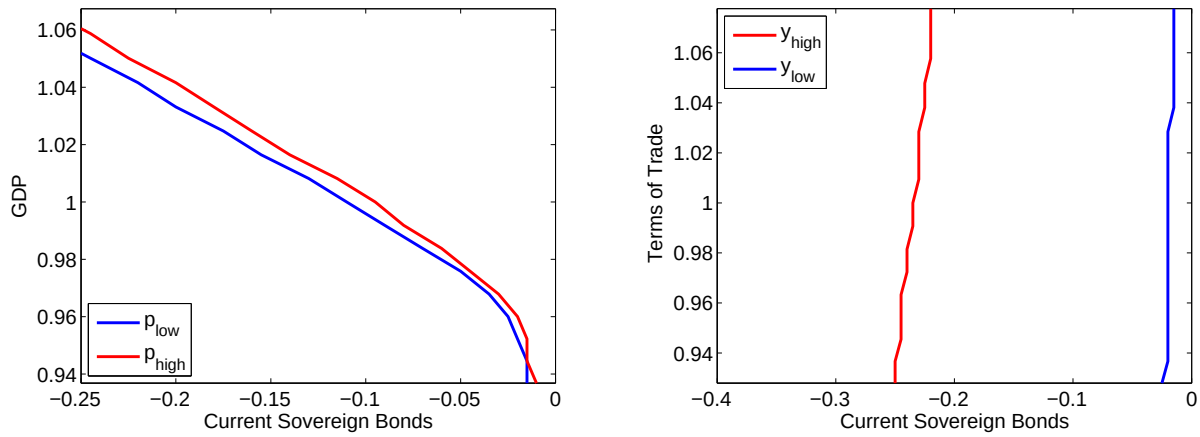


Figure 4 shows the pricing of the sovereign bonds. The left panel shows us how the sovereign bond price schedule decreases in value as the sovereign bond becomes more negative. The right panel shows the implied interest rate using a the domain of sovereign bonds in which the sovereign bond pricing is strictly positive. These graphs deliver two important features. Firstly, this model is able to create countercyclical interest rates due to the link between probability of default and GDP levels. Secondly, the movements of terms of trade increase in importance for sovereign bond price schedule as GDP increases. Moreover, this rise in the price schedule dispersion increases the sovereign intrest rates possibilities. This result helps us explain the movements of spreads during periods where GDP is not below trend. Furthermore, it suggests that terms of trade matter when analyzing interest rates and default likelihoods of emerging economies. This is a step towards explaining the puzzle shown in Tomz and Wright (2007).

Table 6 shows the business cycle statistics of the model. The business cycles is able to recover a couple of the statistics shown in Table 4. But in general, the experiment fails because it is not able to match most of the moments in the data. Despite of this, it is an important step towards understanding the flaws and improving the model.

6 Conclusion

We propose a stochastic general equilibrium model of sovereign default with endogenous default risk in order to explain the interest rate behavior in emerging economies. We incorporate two types of shocks to cover a foreign and a domestic uncertainty. We define as the domestic and the foreign uncertainty, GDP and terms of trade shock, respectively. The

Figure 4: Sovereign Bond Pricing and Sovereign Interest Rate

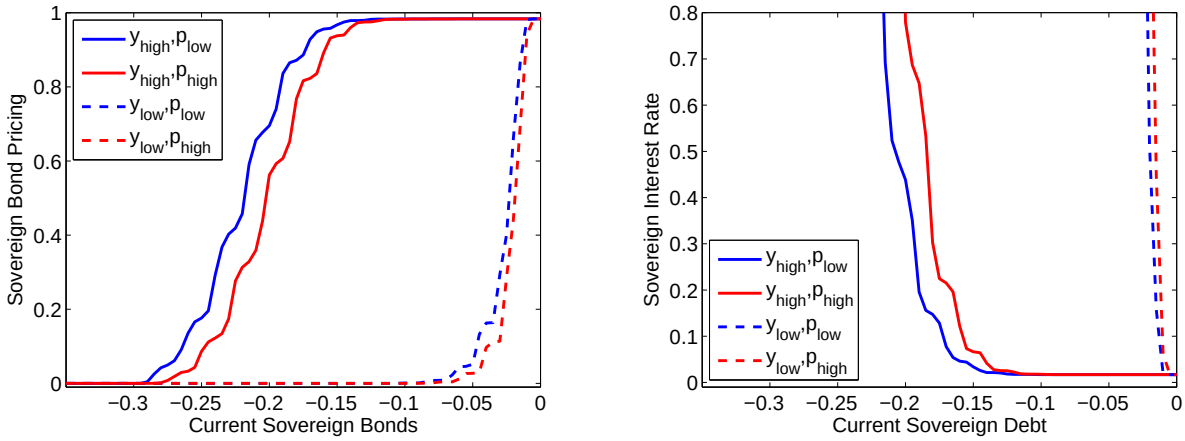


Table 6: Model Business Cycles Statistics

Variable	μ	σ	$\rho(\cdot,\text{Spread})$	$\rho(\cdot,\text{GDP})$	$\rho(\cdot,\text{TOT})$
Spread	0.65%	0.74%	-	-0.2009	0.2567
GDP	-	2.24%	-0.2009	-	-0.2388
Terms of Trade	-	2.52%	0.2567	-0.2388	-
Consumption	-	2.01%	-0.3073	0.9386	-0.2376
Trade Balance	0.00%	0.09%	0.3788	-0.2628	-0.0385
Default probability	2.46%	-	-	-	-
Debt-GDP ratio	3.05%	-	-	-	-

model is able to succesfully increase the dispersion of sovereign interest rates when GDP shocks are above the trend. This result seems to suggest that terms of trade is a good candidate to explain the volatility of interest rates in small open economies when they are not under recessions or crises.

Unfortunately, our business cycles exercise have room for improvements. Nevertheless, the results presented is a great step to explain the behavior of interest rates in emerging economies. Below, we present three issues we are currently working in order to improve our line of research.

Firstly, the VAR(1) process does not capture correctly the dynamics between GDP and terms of trade shown in the data. This can be shown by the small correlation between their contemporaneous realizations in the simulation process. In particular, we are confident that $\mathbb{E}\left[\varepsilon_t^y | \ln p_t\right] \neq 0$, making our estimators in A biased. Moreover, we are not implementing completely the exogeneity assumption of terms of trade. Specifically, we let the future

realization of terms of trade be affected by the current level of GDP. Furhter work of the model will consider improvements in the VAR(1) process presented in (1), for example

$$\begin{bmatrix} \ln y_{t+1} \\ \ln p_{t+1} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \begin{bmatrix} \ln y_t \\ \ln p_t \end{bmatrix} + \begin{bmatrix} 1 & b_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon^y_{t-1} \\ \varepsilon^p_{t=1} \end{bmatrix}.$$

Another weakness of our results is our calibration strategy. We let the tolerance in the algorithm to be high for a faster convergence. The tolerance provided is of $1.0e^{-}1$, which yields really loose results. Moreover, we only do one simulation process in order to compute the target statistics of default probability and standard deviation of trade balance over GDP. Future improvements will encompass a more serious calibration process with a higher tolerance and number of simulation processes. Moreover, we will include as part of it the probability of re-entry to financial markets by targeting the ratio of debt over GDP.

Finally, our computation of the equilibrium can improve greatly. Unfortunately, having three state variables increases greatly the computational cost in terms of time. Because of this we have coarse grids that might miss important movements. Furthermore, as noted by Hatchondo, Martinez and Sapriza (2010), grid search methods can yield spurious results in the business cycles statitics of the model. Future improvements will work on this by imposing finer grids in the state variables. In addition, we will move from the grid search method to methods that are able to capture movements between the grid elements[23].

[23]There are a great variety of methods that allow movements inside the grid elements. In particular, we are currently working on linear and quadratic interpolation methods. We are also interested in implementing innovative methods that have shown effiency in solving these types of models. Specifically, we are interested in the alogorithms provided in McGrattan (1996) and Gordon and Qiu (2015).

References

Aguiar, Mark and Gopinath, Guita. (2007), *The role of interest rates and productivity shocks in emerging market fluctuations.*

Alessandro, Mauro, Sandleris, Guido and Van Der Ghote, Alejandro. (2011). 'Sovereign defaults and the political economy of market reaccess'.

Arellano, Cristina. (2008). 'Default risk and income fluctuations in emerging economies', *The American Economic Review* 98(3), 690–712.

Asonuma, Mr Tamon. (2016), *Sovereign defaults, external debt, and real exchange rate dynamics*, International Monetary Fund.

Broda, Christian. (2004). 'Terms of trade and exchange rate regimes in developing countries', *Journal of International Economics* 63(1), 31–58.

Chatterjee, Satyajit, Corbae, Dean, Nakajima, Makoto and Ríos-Rull, José-Víctor. (2007). 'A quantitative theory of unsecured consumer credit with risk of default', *Econometrica* 75(6), 1525–1589.

Chatterjee, Satyajit and Eyigungor, Burcu. (2012). 'Maturity, indebtedness, and default risk', *The American Economic Review* 102(6), 2674–2699.

Cole, Harold and Kehoe, Timothy. (1996). 'A self-fulfilling model of Mexico's 1994–1995 debt crisis', *Journal of International Economics* 41(3), 309–330.

Cuadra, Gabriel and Nuguer, Victoria. (2016). 'Risky banks and macroprudential policy for emerging economies'.

Cuadra, Gabriel and Sapriza, Horacio. (2006). 'Sovereign default, terms of trade, and interest rates in emerging markets'.

Du, Wenxin and Schreger, Jesse. (2015). 'Sovereign risk, currency risk, and corporate balance sheets', *Harvard, Department of Economics Working Paper* .

Eaton, Jonathan and Gersovitz, Mark. (1981). 'Debt with potential repudiation: Theoretical and empirical analysis', *The Review of Economic Studies* 48(2), 289–309.

Eichengreen, Barry and Hausmann, Ricardo. (1999). 'Exchange rates and financial fragility'.

Gelos, Gaston, Sahay, Ratna and Sandleris, Guido. (2011). 'Sovereign borrowing by developing countries: What determines market access?', *Journal of international Economics* 83(2), 243–254.

Gordon, Grey and Qiu, Shi. (2015). 'A divide and conquer algorithm for exploiting policy function monotonicity', *SSRN 2556345* .

Gu, Grace. (2015). 'A tale of two countries: sovereign default, exchange rate, and trade'.

Hatchondo, Juan Carlos and Martinez, Leonardo. (2009). 'Long-duration bonds and sovereign defaults', *Journal of International Economics* 79(1), 117–125.

Hatchondo, Juan Carlos, Martinez, Leonardo and Sapriza, Horacio. (2010). 'Quantitative properties of sovereign default models: solution methods matter', *Review of Economic dynamics* 13(4), 919–933.

Jeanne, Olivier. (2003), *Why do emerging economies borrow in foreign currency?*, number 3-177, International Monetary Fund.

Kehoe, Timothy and Ruhl, Kim. (2008). 'Are shocks to the terms of trade shocks to productivity?', *Review of Economic Dynamics* 11(4), 804–819.

Kose, Ayhan. (2002). 'Explaining business cycles in small open economies:?How much do world prices matter??', *Journal of International Economics* 56(2), 299–327.

Kose, Ayhan, Towe, Christopher and Meredith, Guy. (2004), *How has NAFTA affected the Mexican economy? Review and evidence*, number 4-59, International Monetary Fund.

McGrattan, Ellen R. (1996). 'Solving the stochastic growth model with a finite element method', *Journal of Economic Dynamics and Control* 20(1), 19–42.

Mendoza, Enrique. (1995). 'The terms of trade, the real exchange rate, and economic fluctuations', *International Economic Review* pp. 101–137.

Mendoza, Enrique and Yue, Vivian. (2011). 'A general equilibrium model of sovereign default and business cycles'.

Neumeyer, Pablo and Perri, Fabrizio. (2005). 'Business cycles in emerging economies: the role of interest rates', *Journal of Monetary Economics* 52(2), 345–380.

Popov, Sergey V, Wiczer, David et al. (2014). 'Equilibrium Sovereign Default with Exchange Rate Depreciation', *Federal Reserve Bank of St. Louis Working Paper Series* (2014-049).

Reinhart, Carmen and Rogoff, Kenneth. (2009), *This time is different: eight centuries of financial folly*, Princeton University Press.

Ruhl, Kims. (2008). 'The international elasticity puzzle', *NYU (Unpublished)* .

Tauchen, George and Hussey, Robert. (1991). 'Quadrature-based methods for obtaining approximate solutions to nonlinear asset pricing models', *Econometrica: Journal of the Econometric Society* pp. 371–396.

Tomz, Michael and Wright, Mark. (2007). 'Do countries default in ?bad times??', *Journal of the European Economic Association* 5(2-3), 352–360.

A Propositions Proofs

A.1 Proposition 3 (Recursive Equilibrium Isomorphism)

Proof Let us use this intratemporal condition (9) and the household's final consumption aggregator,

$$\begin{aligned}
C &= \left(\lambda c_d^{-\eta} + (1-\lambda) c_f^{-\eta}\right)^{-\frac{1}{\eta}} \\
&= c_d \left(\lambda \left(1 + \left(\frac{1-\lambda}{\lambda}\right)\left(\frac{c_d}{c_f}\right)^{\eta}\right)\right)^{-\frac{1}{\eta}} \\
&= c_d \left(\lambda \left(1 + \left(\frac{1-\lambda}{\lambda}\right)\left(\left(\frac{\lambda}{1-\lambda}\right) p\right)^{\frac{\eta}{1+\eta}}\right)\right)^{-\frac{1}{\eta}} \\
&= c_d \left(\lambda \left(1 + \left(\frac{1-\lambda}{\lambda}\right)^{\frac{1}{1+\eta}} p^{\frac{\eta}{1+\eta}}\right)\right)^{-\frac{1}{\eta}} \\
&= \left(\frac{c_d}{\lambda^{\frac{1}{1+\eta}}}\right)\left(\lambda^{\frac{1}{1+\eta}} + (1-\lambda)^{\frac{1}{1+\eta}} p^{\frac{\eta}{1+\eta}}\right)^{-\frac{1}{\eta}}.
\end{aligned}$$

Using the final consumption good price index (10), we can rewrite the previous expression as

$$\frac{c_d^{1+\eta}}{\lambda} = \mathcal{P}(p) C^{1+\eta}. \tag{14}$$

Using the the intratemporal condition (9) and mixing the budget and the balanced trade constraints of (4) and (5), we can realise that the consumption of domestic and foreign expenditure can be expressed as

$$\begin{aligned}
c_d + p c_f &= c_d + \left(\frac{1-\lambda}{\lambda}\right)\left(\frac{c_d}{c_f}\right)^{1+\eta} c_f \\
&= c_d + \left(\frac{1-\lambda}{\lambda}\right) c_d^{1+\eta} c_f^{-\eta} \\
&= \left(\frac{c_d^{1+\eta}}{\lambda}\right)\left(\lambda c_d^{-\eta} + (1-\lambda) c_f^{-\eta}\right) \\
&= \frac{c_d^{1+\eta}}{\lambda C^{\eta}}.
\end{aligned}$$

Thus, using (14) we reach the expression that the household's consumption expenditure can be expressed in terms of the final consumption good and the final consumption good price index (10),

$$c_d + p c_f = \mathcal{P}(p) C.$$

Furthermore, using (9) and (14), we can construct how the consumption of domestic and foreign goods can be decomposed from the aggregate final consumption good as

$$c_d = (\lambda \mathcal{P}(p))^{\frac{1}{1+\eta}} C \qquad \text{and} \qquad c_f = \left(\frac{(1-\lambda)\mathcal{P}(p)}{p} \right)^{\frac{1}{1+\eta}} C.$$

Finally, realise that the restrictions of the maximization problems (4) and (5) are the same as the ones described in (11) and (12). ▬

A.2 Proposition 4 (Default Sets Monotonicity)

Proof Set a level of sovereign debt B_1 and levels of output and terms of trade such that $(y,p) \in \mathcal{D}(B_1)$. Then, it follows that $V^D(y,p) > V^R(B_1,y,p)$. Pick an arbitrary level of sovereign debt B_2 such that $B_2 \leq B_1$. Let us study the resource constraint in (11). Define the budget set of the government contingent to the amount of sovereign bonds due as

$$\mathcal{B}(B) = \{(C,B') \in \mathbb{R}_+ \times [\mathbb{B},\infty) : \quad P(p)C + pq(B',y,p) \leq y + pB\}.$$

By construction, acknowledge that $\mathcal{B}(B_2) \subseteq \mathcal{B}(B_1)$. Because the government is maximizing over a subset of a set, it follows that $V^R(B_1,y,p) \geq V^R(B_2,y,p)$. Joining the inequalities, we conclude that

$$V^D(y,p) > V^R(B_1,y,p) \geq V^R(B_2,y,p).$$

In other words, $(y,p) \in \mathcal{D}(B_2)$. Finally, because the levels of sovereign debt B_2 was taken arbitrarily, we can conclude that $\mathcal{D}(B_1) \subseteq \mathcal{D}(B_2)$. ▬

A.3 Proposition 5 (No Resources Inflows)

Proof Pick an arbitrary $(y,p) \in \mathcal{D}(B)$ and realise this implies $V^D(y,p) > V^R(B,y,p)$. Acknowledge that the resource constraints found in (11) and in (12) can be rewritten respectively as

$$C = \frac{y}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B')B' - B\right) \qquad \text{and} \qquad C = \frac{y}{\mathcal{P}(p)}.$$

Therefore,

$$\begin{aligned} u\left(\frac{y}{\mathcal{P}(p)}\right) + \beta\mathbb{E}_{(y',p')}\left[V^D(y',p')\right] &> \operatorname*{Max}_{B'}\left\{u\left(\frac{y-p(q(B')B'-B)}{\mathcal{P}(p)}\right) + \beta\mathbb{E}_{(y',p')}\left[V(y',p')\right]\right\} \\ &\geq \operatorname*{Max}_{B'}\left\{u\left(\frac{y-p(q(B')B'-B)}{\mathcal{P}(p)}\right) + \beta\mathbb{E}_{(y',p')}\left[V^D(y',p')\right]\right\} \\ &\geq u\left(\frac{y-p(q(B')B'-B)}{\mathcal{P}(p)}\right) + \beta\mathbb{E}_{(y',p')}\left[V^D(y',p')\right], \end{aligned}$$

for all feasible B'. Thus,

$$u\left(\frac{y}{\mathcal{P}(p)}\right) > u\left(\frac{y}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B')B' - B\right)\right).$$

Because $u(\cdot)$ is an increasing function,

$$\frac{y}{\mathcal{P}(p)} \geq \frac{y}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B')B' - B\right).$$

Then, we arrive to

$$q(B')B' - B \geq 0.$$

Finally, because we picked an arbitrary $(y, p) \in \mathcal{D}(B)$, we can conclude that for all feasible B' there are no resources inflows, $q(B')B' - B \geq 0$. ∎

A.4 Proposition 6 (GDP Default Incentives)

Proof Set a level of GDP, terms of trade and sovereign bonds such that $(y_1, p) \in \mathcal{D}(B) \neq \emptyset$. Then, it follows that $V^D(y_1, p) > V^R(B, y_1, p)$. Pick an arbitrary level of GDP y_2 such that $y_2 \leq y_1$. Acknowledge that the resource constraints found in (11) and in (12) can be rewritten respectively as

$$C = \frac{y}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B')B' - B\right) \qquad \text{and} \qquad C = \frac{y}{\mathcal{P}(p)}.$$

To make the proof easier, call

$$B_1' = \arg\operatorname*{Max}_{B'}\left\{u\left(\frac{y_1}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B_1')B_1' - B\right)\right) + \beta\mathbb{E}_{(y',p')}\left[V(B_1', y', p')\right]\right\} \qquad \text{and}$$

$$B_2' = \arg\operatorname*{Max}_{B'}\left\{u\left(\frac{y_2}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B_2')B_2' - B\right)\right) + \beta\mathbb{E}_{(y',p')}\left[V(B_2', y', p')\right]\right\}.$$

Realise that, in particular, these expressions imply,

$$u\left(\frac{y_1}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B_1')B_1' - B\right)\right) + \beta\mathbb{E}_{(y',p')}\left[V(B_1', y', p')\right] \geq$$
$$u\left(\frac{y_1}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B_2')B_2' - B\right)\right) + \beta\mathbb{E}_{(y',p')}\left[V(B_2', y', p')\right].$$

Using Proposition 5 and because $\mathcal{D}(B) \neq \emptyset$ and $u(\cdot)$ is increasing, it follows that $q(B')B' - B \geq 0$ for every feasible B'. In particular, for the optimal level of sovereign bonds using y_2 level of GDP, $q(B_2')B_2' - B \geq 0$.

Because $u(\cdot)$ is strictly concave and $q(B_2')B_2' - B \geq 0$, we know that

$$u\left(\frac{y_1}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B_2')B_2' - B\right)\right) - u\left(\frac{y_2}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B_2')B_2' - B\right)\right) \geq u\left(\frac{y_1}{\mathcal{P}(p)}\right) - u\left(\frac{y_1}{\mathcal{P}(p)}\right).$$

Moreover, the right hand side can be expressed as

$$\begin{aligned} u\left(\frac{y_1}{\mathcal{P}(p)}\right) - u\left(\frac{y_2}{\mathcal{P}(p)}\right) &= \left(u\left(\frac{y_1}{\mathcal{P}(p)}\right) + \beta\mathbb{E}_{(y',p')}\left[V(y',p')\right]\right) - \left(u\left(\frac{y_2}{\mathcal{P}(p)}\right) + \beta\mathbb{E}_{(y',p')}\left[V(y',p')\right]\right) \\ &= V^D(y_1,p) - V^D(y_2,p). \end{aligned}$$

Therefore,

$$\begin{aligned} V^D(y_1,p) - V^D(y_2,p) &\leq u\left(\frac{y_1}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B_2')B_2' - B\right)\right) - u\left(\frac{y_2}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B_2')B_2' - B\right)\right) \\ &= \left(u\left(\frac{y_1}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B_2')B_2' - B\right)\right) + \beta\mathbb{E}_{(y',p')}\left[V(B_2',y',p')\right]\right) \\ &\quad - \left(u\left(\frac{y_2}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B_2')B_2' - B\right)\right) + \beta\mathbb{E}_{(y',p')}\left[V(B_2',y',p')\right]\right) \\ &\leq \left(u\left(\frac{y_1}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B_1')B_1' - B\right)\right) + \beta\mathbb{E}_{(y',p')}\left[V(B_1',y',p')\right]\right) \\ &\quad - \left(u\left(\frac{y_2}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B_2')B_2' - B\right)\right) + \beta\mathbb{E}_{(y',p')}\left[V(B_2',y',p')\right]\right) \\ &= V^R(B,y_1,p) - V^R(B,y_2,p) \\ &< V^D(y_1,p) - V^R(B,y_2,p). \end{aligned}$$

Then we arrive to following expression $V^D(y_2,p) > V^R(B,y_2,p)$. Finally we can conclude, because the level of GDP y_2 was taken arbitrarily, we can conclude that $(y_2,p) \in \mathcal{D}(B)$.

A.5 Proposition 7 (Terms of Trade Default Incentives)

Proof Set a level of GDP, terms of trade and sovereign bonds such that $(y,p_1) \in \mathcal{D}(B) \neq \emptyset$. Then, it follows that $V^D(y,p_1) > V^R(B,y,p_1)$. Pick an arbitrary level of terms of trade p_2 such that $p_2 \geq p_1$. Acknowledge that the resource constraints found in (11) and in (12) can be rewritten respectively as

$$C = \frac{y}{\mathcal{P}(p)} - \frac{p}{\mathcal{P}(p)}\left(q(B')B' - B\right) \qquad \text{and} \qquad C = \frac{y}{\mathcal{P}(p)}.$$

To make the proof easier, call

$$B_1' = \arg\underset{B'}{\text{Max}} \left\{ u\left(\frac{y}{\mathcal{P}(p_1)} - \frac{p_1}{\mathcal{P}(p_1)} \left(q(B_1')B_1' - B \right) \right) + \beta \mathbb{E}_{(y',p')} \left[V(B_1', y', p') \right] \right\} \qquad \text{and}$$

$$B_2' = \arg\underset{B'}{\text{Max}} \left\{ u\left(\frac{y}{\mathcal{P}(p_2)} - \frac{p_2}{\mathcal{P}(p_2)} \left(q(B_2')B_2' - B \right) \right) + \beta \mathbb{E}_{(y',p')} \left[V(B_2', y', p') \right] \right\}.$$

Realise that, in particular, these expressions imply,

$$u\left(\frac{y}{\mathcal{P}(p_1)} - \frac{p_1}{\mathcal{P}(p_1)} \left(q(B_1')B_1' - B \right) \right) + \beta \mathbb{E}_{(y',p')} \left[V(B_1', y', p') \right] \geq$$
$$u\left(\frac{y}{\mathcal{P}(p_1)} - \frac{p_1}{\mathcal{P}(p_1)} \left(q(B_2')B_2' - B \right) \right) + \beta \mathbb{E}_{(y',p')} \left[V(B_2', y', p') \right].$$

Using Proposition 5 and because $\mathcal{D}(B) \neq \emptyset$ and $u(\cdot)$ is increasing, it follows that $q(B')B' - B \geq 0$ for every feasible B'. In particular, for the optimal level of sovereign bonds using y_2 level of GDP, $q(B_2')B_2' - B \geq 0$.

Acknowledge that the final consumption good price index is increasing,

$$\mathcal{P}'(p) = \left(\frac{(1-\lambda)\mathcal{P}(p)}{p} \right)^{\frac{1}{1+\eta}} >\geq 0.$$

Therefore, the ratio $\frac{p}{\mathcal{P}(p)}$ is increasing too,

$$\begin{aligned}
\frac{d}{dp}\left(\frac{p}{\mathcal{P}(p)} \right) &= \frac{\mathcal{P}(p) - p\mathcal{P}'(p)}{\left(\mathcal{P}(p)\right)^2} \\
&= \frac{1 - (1-\lambda)^{\frac{1}{1+\eta}} p^{\frac{\eta}{1+\eta}} \left(\mathcal{P}(p)\right)^{-\frac{\eta}{1+\eta}}}{\mathcal{P}(p)} \\
&= \frac{\left(\mathcal{P}(p)\right)^{\frac{\eta}{1+\eta}} - (1-\lambda)^{\frac{1}{1+\eta}} p^{\frac{\eta}{1+\eta}}}{\left(\mathcal{P}(p)\right)^{\frac{1+2\eta}{1+\eta}}} \\
&= \left(\frac{\lambda}{\left(\mathcal{P}(p)\right)^{1+2\eta}} \right)^{\frac{1}{1+\eta}} \\
&\geq 0.
\end{aligned}$$

In other words, it follows that $\frac{p_2}{\mathcal{P}(p_2)} \geq \frac{p_1}{\mathcal{P}(p_1)}$ because $p_2 \geq p_1$.

Using the previous result and because $u(\cdot)$ is strictly concave and $q(B_2')B_2' - B \geq 0$, we know that

$$\begin{aligned}
u\left(\frac{y}{\mathcal{P}(p_1)} \right) - u\left(\frac{y}{\mathcal{P}(p_2)} \right) &\leq u\left(\frac{y}{\mathcal{P}(p_1)} - \frac{p_2}{\mathcal{P}(p_2)} \left(q(B_2')B_2' - B \right) \right) - u\left(\frac{y}{\mathcal{P}(p_2)} - \frac{p_2}{\mathcal{P}(p_2)} \left(q(B_2')B_2' - B \right) \right) \\
&\leq u\left(\frac{y}{\mathcal{P}(p_1)} - \frac{p_1}{\mathcal{P}(p_1)} \left(q(B_2')B_2' - B \right) \right) - u\left(\frac{y}{\mathcal{P}(p_2)} - \frac{p_2}{\mathcal{P}(p_2)} \left(q(B_2')B_2' - B \right) \right)
\end{aligned}$$

Moreover, the left hand side can be expressed as

$$u\left(\frac{y}{\mathcal{P}(p_1)}\right) - u\left(\frac{y}{\mathcal{P}(p_2)}\right) = \left(u\left(\frac{y}{\mathcal{P}(p_1)}\right) + \beta\mathbb{E}_{(y',p')}\left[V(y',p')\right]\right) - \left(u\left(\frac{y}{\mathcal{P}(p_2)}\right) + \beta\mathbb{E}_{(y',p')}\left[V(y',p')\right]\right)$$
$$= V^D(y,p_1) - V^D(y,p_2).$$

Therefore,

$$\begin{aligned}
V^D(y,p_1) - V^D(y,p_2) &\leq u\left(\frac{y}{\mathcal{P}(p_1)} - \frac{p_1}{\mathcal{P}(p_1)}\left(q(B_2')B_2' - B\right)\right) - u\left(\frac{y}{\mathcal{P}(p_2)} - \frac{p_2}{\mathcal{P}(p_2)}\left(q(B_2')B_2' - B\right)\right) \\
&= \left(u\left(\frac{y}{\mathcal{P}(p_1)} - \frac{p_1}{\mathcal{P}(p_1)}\left(q(B_2')B_2' - B\right)\right) + \beta\mathbb{E}_{(y',p')}\left[V(B_2',y',p')\right]\right) \\
&\quad - \left(u\left(\frac{y}{\mathcal{P}(p_2)} - \frac{p_2}{\mathcal{P}(p_2)}\left(q(B_2')B_2' - B\right)\right) + \beta\mathbb{E}_{(y',p')}\left[V(B_2',y',p')\right]\right) \\
&\leq \left(u\left(\frac{y}{\mathcal{P}(p_1)} - \frac{p_1}{\mathcal{P}(p_1)}\left(q(B_1')B_1' - B\right)\right) + \beta\mathbb{E}_{(y',p')}\left[V(B_1',y',p')\right]\right) \\
&\quad - \left(u\left(\frac{y}{\mathcal{P}(p_2)} - \frac{p_2}{\mathcal{P}(p)}\left(q(B_2')B_2' - B\right)\right) + \beta\mathbb{E}_{(y',p')}\left[V(B_2',y',p')\right]\right) \\
&= V^R(B,y,p_1) - V^R(B,y,p_2) \\
&< V^D(y,p_1) - V^R(B,y,p_2).
\end{aligned}$$

Then we arrive to following expression $V^D(y,p_2) > V^R(B,y,p_2)$. Finally we can conclude, because the level of terms of trade p_2 was taken arbitrarily, we can conclude that $(y,p_2) \in \mathcal{D}(B)$. ∎

B Computational Algorithm

We extend the algoritm described in Arellano (2008) and incorporate the one loop enhancement proposed in Hatchondo, Martinez and Sapriza (2010) using a grid search method. The following is the algorithm we follow to solve the model proposed in Definition 2 and the calibration strategy for the discount factor β and the GDP default penalty parameter κ:

1) Fix the calibration targets of default probability $\bar{d}$ and standard deviation of trade balance over GDP $\bar{s}$.

2) Discretize B space and dicretize (y,p) space using Tauchen and Hussey (1991)

3) Propose a guess for the discount factor β and the GDP default penalty parameter κ,

$$\beta = 0.95 \qquad \text{and} \kappa = 1.00$$

4) Propose a guess for the set of value functions V, V^R, V^D and the sovereign bond price schedule q,

$$V = [0], \qquad V^R = [0], \qquad V^D = [0], \qquad \text{and} \qquad q = \left[\frac{1}{1+r^*}\right].$$

5) For every state (B, y, p), solve the repayment state maximization problem (11) and compute an implied repayment state value function $\hat{V}^R$.

6) For every state (y, p), solve the repayment state maximization problem (12) and compute an implied default state value function $\hat{V}^D$.

7) For every state (B, y, p), solve the maximization problem (**??**) and compute an implied default state value function $\hat{V}$.

8) Using the implied value functions $\hat{V}^R$ and $\hat{V}^D$, construct default set $\mathcal{D}$

9) Using the default set $\mathcal{D}$, construct default probability schedule δ

10) Using the default probability schedule δ, construct an implied sovereign bond price schedule $\hat{q}$

11) Compute the error term as

$$z_1 = ||V - \hat{V}||_\infty + ||V^R - \hat{V}^R||_\infty + ||V^D - \hat{V}^D||_\infty + ||q - \hat{q}||_\infty.$$

12) Update guesses $V = \hat{V}$, $V^R = \hat{V}^R$, $V^D = \hat{V}^D$, and $q = \hat{q}$.

13) If $z_1 \geq 1.0e^{-6}$, return to 5).

14) Construct one simulation of 10,000 periods with 500 burn-ins starting with the zero sovereign bonds and the long-run levels of GDP and terms of trade

15) Construct the implied default probability α and the implied standard deviation of trade balance over GDP s

16) Compute the error term as

$$z_2 = \frac{\left|d - \bar{d}\right| + |s - \bar{s}|}{2}.$$

17) If $z_2 \geq 1.0e^{-1}$, update guesses β and κ, then return to 4)